Representation Parameters for Classical Groups and the Real Weyl Groups

Annegret Paul

Western Michigan University

CBMS UMass Boston
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Overview

Let G be a real reductive Lie group.

Problem

Find all irreducible admissible representations of G.

Tool (as DAV explained in the last lecture):
The Langlands classification says:

Representations correspond to pairs (H, χ), up to conjugation by G.
Here H is a Cartan subgroup (CSG) of G, and χ a character of H.

Understand conjugation by G:

- Find the conjugacy classes of Cartan subgroups (CSG)
- For each CSG H, compute the *real Weyl group* $W(G, H) = N_G(H)/H \cong N_K(H)/(H \cap K)$

Goal: Find $\widehat{H}/W(G,H)$ for all conjugacy classes of CSGs H.

To find all (theta stable) H, we find lots of CSGs, show they are pairwise not conjugate, then show that these are all.

Tool: Kostant's "Cascade construction".

Need:

- Fundamental (maximally compact) Cartan subgroup $H_0 = Cent_G(\mathfrak{h}_0)$
- $\Delta = \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{0,\mathbb{C}})$
- $W(G, H_0)$

CSGs are in bijection with sets of *strongly orthogonal* (i. e., orthogonal, and no sum or difference of two is a root) noncompact imaginary roots, up to conjugation by $W(G, H_0)$.

Example: G=SL(2,R)

$H_0 = K = SO(2)$, $\Delta = \{\pm\alpha\}$, α is noncompact.
K acts trivially on H_0, so $W(G, H_0) = \{1\}$.

Sets of strongly orthogonal roots: $\varnothing$ and $\{\alpha\}$, corresponding to H_0 and $H_1 = \left\{ \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix} \right\} \simeq \mathbb{R}^\times$.

The element $\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in K$ switches the diagonal entries. Since conjugation preserves eigenvalues, this is the only nontrivial Weyl group element. So $W(G, H_1) \simeq \mathbb{Z}/2\mathbb{Z}$, with the nontrivial element acting by inversion on $\mathbb{R}^\times$.

$\widehat{H_0}/W(G, H_0) \longleftrightarrow \mathbb{Z}$, $k \in \mathbb{Z} \Rightarrow \chi_k(e^{i\theta}) = e^{ik\theta}$;

$\widehat{H_1} \longleftrightarrow \mathbb{Z}/2\mathbb{Z} \times \mathbb{C}$, $\chi_{(\epsilon,\nu)}(r) = sgn(r)^\epsilon |r|^\nu$, $(\epsilon, \nu) \sim (\epsilon, -\nu)$.

Example GL(2,R)

$H_0 = T_0 A_0 \simeq \mathbb{C}^\times$, $T_0 = SO(2)$, $A_0 = \left\{ \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} \right\} \simeq \mathbb{R}_+^\times$

$W(G, H_0) = W(K, T_0)$ (since A_0 is in the center of G).
$\sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in O(2)$ acts by inversion. Conjugation preserves eigenvalues and trace, so this is the only nontrivial element.

$W(G, H_0) \simeq \mathbb{Z}/2\mathbb{Z}$.

$\Delta = \{\pm\alpha\}$ noncompact imaginary. As for $SL(2, \mathbb{R})$, we have two CSGs, H_0 and the diagonal split $H_1 \simeq (\mathbb{R}^\times)^2$.
$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in K$ switches the diagonal entries, and that is the only possibility.

$W(G, H_1) \simeq \mathbb{Z}/2\mathbb{Z}$.

GL(2,R), representations attached to the fundamental CSG

$H_0 \cong \mathbb{C}^\times$, so $\widehat{H_0} \longleftrightarrow \mathbb{Z} \times \mathbb{C}$, $\chi_{(k,\nu)}(re^{i\theta}) = r^\nu e^{ik\theta}$.

$W(G, H_0) : (k, \nu) \sim (-k, \nu)$

Which representation does this parametrize?

Real parabolic $P_0 = M_0 A_0 N_0$,

$M_0 = SL(2, \mathbb{R})^\pm$, $k \leftrightarrow$ discrete series or limit of d.s. σ_k of M_0

$\nu \leftrightarrow$ character of A_0

$\pi(k, \nu) =$ Langlands quotient of $Ind_{P_0}^G(\sigma_k \otimes \nu \otimes 1)$.

(This is, of course, just $\sigma_k \otimes \chi_\nu$ in this case, since $G = P_0$ and N_0 is trivial.)

GL(2,R), representations attached to the split CSG

$H_1 \cong (R^\times)^2$, so $\widehat{H_1} \longleftrightarrow (\mathbb{Z}/2\mathbb{Z})^2 \times (\mathbb{C})^2$

$W(G, H_1) : (\epsilon_1, \epsilon_2, \nu_1, \nu_2) \sim (\epsilon_2, \epsilon_1, \nu_2, \nu_1)$

Real parabolic: minimal parabolic $P_1 = T_1 A_1 N_1$,
$T_1 = \left\{ \begin{pmatrix} \eta_1 & 0 \\ 0 & \eta_2 \end{pmatrix} : \eta_i = \pm 1 \right\}$, $A_1 = \left\{ \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix} : r_i \in \mathbb{R}_+^\times \right\}$

$$\chi_{(\epsilon_1,\epsilon_2,\nu_1,\nu_2)} \left(\begin{pmatrix} \eta_1 & 0 \\ 0 & \eta_2 \end{pmatrix} \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix} \right) = \eta_1^{\epsilon_1} \eta_2^{\epsilon_2} r_1^{\nu_1} r_2^{\nu_2}$$

$\pi(\epsilon_1, \epsilon_2, \nu_1, \nu_2)$ = LQ of the minimal principal series
$Ind_{P_1}^G(\chi_{(\epsilon_1,\epsilon_2,\nu_1,\nu_2)} \otimes 1)$

Note: The principal series LQ $\pi\left(1, 0, \frac{\nu}{2}, \frac{\nu}{2}\right)$ coincides with $\pi(0, \nu)$ (attached to H_0) for any ν.

GL(4,R) (towards GL(n,R))

$K = O(4)$ has rank 2, so $T_0 = SO(2)^2$

$A_0 = \left(\mathbb{R}_+^\times\right)^2$, and $H_0 = T_0 A_0 \simeq \left(\mathbb{C}^\times\right)^2$

In $\mathfrak{g}$, $\mathfrak{h}_0 = \left\{ X = \begin{pmatrix} r & x & 0 & 0 \\ -x & r & 0 & 0 \\ 0 & 0 & s & y \\ 0 & 0 & -y & s \end{pmatrix} : r, s, x, y \in \mathbb{R} \right\}$

We also have $H_1 \simeq \mathbb{C}^\times \times \left(\mathbb{R}^\times\right)^2$, and $H_2 \simeq \left(\mathbb{R}^\times\right)^4$,

$$\mathfrak{h}_1 = \left\{ \begin{pmatrix} r & x & 0 & 0 \\ -x & r & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & 0 & 0 & t \end{pmatrix} \right\}, \qquad \mathfrak{h}_2 = \left\{ \begin{pmatrix} r & 0 & 0 & 0 \\ 0 & s & 0 & 0 \\ 0 & 0 & t & 0 \\ 0 & 0 & 0 & u \end{pmatrix} \right\}$$

Are these all?

Let $\varepsilon_1, \varepsilon_2, f_1, f_2 \in \mathfrak{h}_0^*$ be defined by $\varepsilon_1(X) = ix$, $\varepsilon_2(X) = iy$, $f_1(X) = r$, $f_2(X) = s$.

$\Delta = \{\pm 2\varepsilon_1, \pm 2\varepsilon_2, \pm\varepsilon_1 \pm \varepsilon_2 \pm (f_1 - f_2)\}$

$\Delta_i = \{\pm 2\varepsilon_1, \pm 2\varepsilon_2\}$, all noncompact. Clearly, $2\varepsilon_1$ and $2\varepsilon_2$ are strongly orthogonal. Are they conjugate?

It is easy to see that $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ takes $2\varepsilon_1$ to -$2\varepsilon_2$.

Sets of strongly orthogonal nc roots: $\varnothing$, $\{2\varepsilon_1\}$, $\{2\varepsilon_1, 2\varepsilon_2\}$. So we have all CSGs.

GL(4,R) continued

Characters of the CSGs:

$H_0 \simeq (\mathbb{C}^\times)^2$, so $\quad \widehat{H_0} \longleftrightarrow \mathbb{Z}^2 \times \mathbb{C}^2$

$H_1 \simeq \mathbb{C}^\times \times (\mathbb{R}^\times)^2$, so $\quad \widehat{H_1} \longleftrightarrow \mathbb{Z} \times \mathbb{C} \times (\mathbb{Z}/2\mathbb{Z})^2 \times \mathbb{C}^2$

$H_2 \simeq (\mathbb{R}^\times)^4$, so $\quad \widehat{H_2} \longleftrightarrow (\mathbb{Z}/2\mathbb{Z})^4 \times \mathbb{C}^4$

Note: Each set is a 4 parameter family of characters.

What about the real Weyl groups???

Fix a CSG $H = TA$, and let $M = Cent_G(A)$, so that MA is a Levi subgroup of G.

Theorem (Vogan, IC4)

$$\begin{aligned} W(G,H) &\simeq (W_C)^\theta \ltimes (W(M \cap K, H) \times W_r) \\ &\simeq (W_C)^\theta \ltimes ((W_{i,c} \ltimes \mathcal{A}(H)) \times W_r) \end{aligned}$$

- $W_{i,c}$ is the Weyl group of the system $\Delta_{i,c}$ of compact imaginary roots in $\Delta = \Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$;
- W_r is the Weyl group of the system Δ_r of real roots in Δ;
- $\mathcal{A}(H)$ is an abelian two-group which depends on the group G (not just on the Lie algebra and root system); if G is "nice", then $W_{i,c} \ltimes \mathcal{A}(H)$ is contained in the Weyl group W_i of the imaginary roots in Δ. The group $\mathcal{A}(H)$ must preserve $\Delta_{i,c}$.
- Let $\Delta_C = \{\alpha \in \Delta : \alpha \text{ is complex and } (\alpha, \rho_i) = (\alpha, \rho_r) = 0\}$ (a root system). Then W_C is the Weyl group of Δ_C, and $(W_C)^\theta$ the θ-fixed part of it.

GL(4,R), Fundamental CSG

$\Delta_{i,c} = \varnothing = \Delta_r \implies W_{i,c}$ and W_r are trivial.

To compute the abelian 2-group $\mathcal{A}(H_0)$, consider $\Delta_{i,n} = \{\pm 2\varepsilon_1, \pm 2\varepsilon_2\}$. If our general element of $\mathfrak{h}_0$ is written

$$X(x,y,r,s) = \begin{pmatrix} r & x & 0 & 0 \\ -x & r & 0 & 0 \\ 0 & 0 & s & y \\ 0 & 0 & -y & s \end{pmatrix}$$

the two root reflections change the sign on x and on y, resp. They are represented by $diag(1,-1,1,1)$ and $diag(1,1,1,-1)$ in $O(4)$.

So $\mathcal{A}(H_0)$ has rank 2, and order 4; it is isomorphic to W_i.

Note: $diag(1,-1,1,1)$ and $diag(1,1,1,-1)$ are not in $SO(4)$, but their product is. In $SL(4,\mathbb{R})$, $\mathcal{A}(H_0)$ has rank 1 and order 2. (Changing the sign on both coordinates simultaneously.)

The complex piece of the Weyl group

The group $(W_C)^\theta$

Let $\Delta_C = \{\alpha \in \Delta : \alpha$ is complex, and orthogonal to both ρ_i and $\rho_r\}$.
Then Δ_C is a root system, which is the disjoint union of two systems Δ_1 and Δ_2 such that $\Delta_2 = \{\theta(\alpha) : \alpha \in \Delta\}$.

Then $(W_C)^\theta$ is generated by the $\{s_\alpha s_{\theta\alpha} : \alpha \in \Delta_1\}$.

For $GL(4, \mathbb{R})$ and H_0, we can take $\rho_i = \varepsilon_1 + \varepsilon_2$, and $\rho_r = 0$. Then

$$\begin{aligned}\Delta_C &= \{\pm(\varepsilon_1 - \varepsilon_2) \pm (f_1 - f_2)\} \\ &= \{\pm(\varepsilon_1 - \varepsilon_2 + f_1 - f_2)\} \cup \{\pm(\varepsilon_1 - \varepsilon_2 - f_1 + f_2)\}.\end{aligned}$$

Then $(W_C)^\theta = \langle w = s_\alpha s_{\theta\alpha}\rangle$, where $\alpha = \varepsilon_1 - \varepsilon_2 + f_1 - f_2$.

$w \cdot X(x, y, r, s) = X(y, x, s, r)$ (switching the two $\mathbb{C}^\times$ factors)

Parameters for GL(4,R)

$W(G, H_0)$ has order 8. The parameters attached to H_0 are therefore:

$(k_1, k_2, \nu_1, \nu_2) \in \mathbb{Z}^2 \times \mathbb{C}^2$ with
$(k_1, k_2, \nu_1, \nu_2) \sim (\pm k_1, \pm k_2, \nu_1, \nu_2) \sim (k_2, k_1, \nu_2, \nu_1)$.

Note: $M_0 A_0 = GL(2, \mathbb{R}) \times GL(2, \mathbb{R}) \hookrightarrow GL(4, \mathbb{R})$, and each pair (k_i, ν_i) determines a representation of $GL(2, \mathbb{R})$.

For $H_2 \simeq (\mathbb{R}^\times)^4$, all roots are real.

$W(G, H_2) \simeq W_r \simeq W(A_3) \simeq S_4$

The parameters are $(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \nu_1, \nu_2, \nu_3, \nu_4) \in (\mathbb{Z}/2\mathbb{Z})^4 \times \mathbb{C}^4$, up to permutation of the four indices.

GL(4,R), CSG 1

Now for H_1 :

$$X = X(x, r, s, t) := \begin{pmatrix} r & x & 0 & 0 \\ -x & r & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & 0 & 0 & t \end{pmatrix} \in \mathfrak{h}_1$$

Define $\varepsilon(X) = ix$, $f_1(X) = r$, $f_2(X) = s$, $f_3(X) = t$.

$\Delta_i = \Delta_{i,n} = \{\pm 2\varepsilon\}$, $\Delta_r = \{\pm(f_2 - f_3)\}$,
complex roots $\{\pm\varepsilon \pm (f_1 - f_2), \pm\varepsilon \pm (f_1 - f_3)\}$

$W_{i,c}$ is trivial. $\mathcal{A}(H_1)$ has rank 1; it coincides with $W_i = \langle s_{2\varepsilon} \rangle$.

$(W_C)^\theta$ is trivial because no complex root is orthogonal to ρ_i.

$W_r = \langle s_{f_2 - f_3} \rangle$ (switching the two coordinates)

The parameters are $(k, \nu, \epsilon_1, \epsilon_2, \nu_1, \nu_2) \in \mathbb{Z} \times \mathbb{C} \times (\mathbb{Z}/2\mathbb{Z})^2 \times \mathbb{C}^2$ with
$(k, \nu, \epsilon_1, \epsilon_2, \nu_1, \nu_2) \sim (-k, \nu, \epsilon_1, \epsilon_2, \nu_1, \nu_2) \sim (k, \nu, \epsilon_2, \epsilon_1, \nu_2, \nu_1)$.

Checking with atlas: CSGs

Cartan #0:
split: 0; compact: 0; complex: 2
canonical twisted involution: e
twisted involution orbit size: 3; fiber size: 1; strong inv: 3
imaginary root system: A1.A1
real root system is empty
complex factor: A1
real form #1: [0] (1)
real form #0: [1] (1)

Cartan #1:
split: 2; compact: 0; complex: 1
canonical twisted involution: 1,2,3,2,1
twisted involution orbit size: 6; fiber size: 1; strong inv: 6

imaginary root system: A1
real root system: A1
complex factor is empty
real form #1: [0] (1)

Cartan #2:
split: 4; compact: 0; complex: 0
canonical twisted involution: 1,2,1,3,2,1
twisted involution orbit size: 1; fiber size: 1; strong inv: 1
imaginary root system is empty
real root system: A3
complex factor is empty
real form #1: [0] (1)

real:

Weyl Groups; Fundamental CSG

real: realweyl
choose Cartan class (one of 0,1,2): 0
Name an output file (return for stdout, ? to abandon):

real weyl group is $W^C.((A.W_{ic})xW^R)$, where:
W^C is isomorphic to a Weyl group of type A1
A is an elementary abelian 2-group of rank 2
W_{ic} is trivial
W^R is trivial

generators for W^C:
1,3
generators for A
2
1,2,3,2,1

real:

Weyl Group for CSG1

real: realweyl
choose Cartan class (one of 0,1,2): 1
Name an output file (return for stdout, ? to abandon):

real weyl group is $W^C.((A.W_{ic})xW^R)$, where:
W^C is trivial
A is an elementary abelian 2-group of rank 1
W_{ic} is trivial
W^R is a Weyl group of type A1

generators for A
2
generators for W^R:
1,2,3,2,1

real:

Weyl group for the split CSG

real: realweyl
choose Cartan class (one of 0,1,2): 2
Name an output file (return for stdout, ? to abandon):

real weyl group is $W^C.((A.W_{ic})xW^R)$, where:
W^C is trivial
A is trivial
W_{ic} is trivial
W^R is a Weyl group of type A3

generators for W^R:
1
2
3

real:

GL(n,R)

Let $m = \lfloor \frac{n}{2} \rfloor$, $\delta = n - 2m = 0$ or 1.

Fundamental CSG: $H_0 \simeq (\mathbb{C}^\times)^m \times (\mathbb{R}^\times)^\delta$

There are $m+1$ CSGs: $H_j \simeq (\mathbb{C}^\times)^j \times (\mathbb{R}^\times)^{n-2j}$ for $0 \leq j \leq m$.

$\widehat{H_j} \longleftrightarrow \mathbb{Z}^j \times \mathbb{C}^j \times (\mathbb{Z}/2\mathbb{Z})^{n-2j} \times \mathbb{C}^{n-2j}$

Define a basis of $\mathfrak{h}_j^*$ analogous to before: ε_i and f_i for $1 \leq i \leq j$, and g_i for $1 \leq i \leq n-2j$.

Imaginary roots (all noncompact): $\pm 2\varepsilon_i$ for $1 \leq i \leq j$.

This gives a root system of type $(A_1)^j$. $W_{i,c} = \{1\}$, W_i is an abelian 2-group of rank j.

Each reflection is represented by a (diagonal) element in $O(n)$, so $\mathcal{A}(H_j) \simeq (\mathbb{Z}/2\mathbb{Z})^j$, inverting the elements of the $SO(2)$ factors.

GL(n,R) continued

Real roots: $\Delta_r = \{\pm(g_i - g_l) : 1 \leq i < l \leq n - 2j\}$

$W_r \simeq S_{2n-j}$, permuting the split factors.

Complex roots: $\{\pm\varepsilon_i \pm \varepsilon_l \pm (f_i - f_l)\} \cup \{\pm\varepsilon_i \pm (f_i - g_l)\}$

Choose $\rho_i = \sum \varepsilon_i$, and ρ_r arbitrary, then $\Delta_C = \{\pm(\varepsilon_i - \varepsilon_l) \pm (f_i - f_l)\}$, a root system of type $A_{j-1} \times A_{j-1}$.

$(W_C)^\theta \simeq S_j$ acts by permuting the $\mathbb{C}^\times$ factors.

Lemma

$\widehat{H_j} / W(G, H_j) \longleftrightarrow \{(k_1, \ldots, k_j, \nu_1, \ldots, \nu_j, \epsilon_1, \ldots, \epsilon_{n-2j}, \mu_1, \ldots, \mu_{n-2j})\} / \sim$
$k_i \in \mathbb{Z}$, $\nu_i, \mu_i \in \mathbb{C}$, $\epsilon_i \in \mathbb{Z}/2\mathbb{Z}$, modulo:
permutation of the first j and the last $n - 2j$ coordinates, and modulo sign on the k_i.

For each j, this is an n-parameter family of characters.

New Example: G=SO(2,2)

$K = S(O(2) \times O(2)), \quad H_0 \simeq SO(2)^2$

$$\mathfrak{h}_0 = \left\{ X(x,y) = \begin{pmatrix} 0 & x & 0 & 0 \\ -x & 0 & 0 & 0 \\ 0 & 0 & 0 & y \\ 0 & 0 & -y & 0 \end{pmatrix} \right\};$$

Define $\varepsilon_1\left(X(x,y)\right) = ix$, $\varepsilon_2\left(X(x,y)\right) = iy$.

Roots: $\Delta = \Delta_i = \{\pm\varepsilon_1 \pm \varepsilon_2\}$ all noncompact, Type $A_1 \times A_1$.

$\varepsilon_1 - \varepsilon_2$ and $\varepsilon_1 + \varepsilon_2$ are strongly orthogonal;

Are they conjugate by $W(G, H_0) = W_{i,c} \ltimes \mathcal{A}(H_0) = \mathcal{A}(H_0)$?

Recall that W_i acts only by switching the coordinates, and/or changing both signs.
Since $\mathcal{A}(H_0) \subset W_i$, the two roots are NOT conjugate.

SO(2,2) continued

Notice: $diag(1, -1, 1, -1)$ changes both signs, but no element of K switches the two entries.

$\mathcal{A}(H_0) = \langle s_{\varepsilon_1 - \varepsilon_2} s_{\varepsilon_1 + \varepsilon_2} \rangle \simeq \mathbb{Z}/2\mathbb{Z}$

4 strongly orthogonal sets: $\varnothing$, $\{\varepsilon_1 - \varepsilon_2\}$, $\{\varepsilon_1 + \varepsilon_2\}$, $\{\varepsilon_1 - \varepsilon_2, \varepsilon_1 + \varepsilon_2\}$.

Recall: Embedding $GL(n, \mathbb{R}) \hookrightarrow SO(n, n)$

Cartan (polar) decomposition: $A \in \mathfrak{gl}(n, \mathbb{R})$, $A = U + P$ with $U \in \mathfrak{so}(n)$, P symmetric.

Then $A \longmapsto \begin{pmatrix} U & P \\ P & U \end{pmatrix} \in \mathfrak{so}(\mathfrak{n}, \mathfrak{n})$, and this is a Lie algebra map.

The CSGs of $GL(n, \mathbb{R})$ then map to CSGs of $SO(n, n)$ (since both have rank n).

CSGs of SO(2,2)

$$H_1 \simeq \mathbb{C}^\times,\ \mathfrak{h}_1 = \left\{ X(x,r) = \begin{pmatrix} 0 & x & r & 0 \\ -x & 0 & 0 & r \\ r & 0 & 0 & x \\ 0 & r & -x & 0 \end{pmatrix} \right\}$$

$$H_2 \simeq (\mathbb{R}^\times)^2,\ \mathfrak{h}_2 = \left\{ X(r,s) = \begin{pmatrix} 0 & 0 & r & 0 \\ 0 & 0 & 0 & s \\ r & 0 & 0 & 0 \\ 0 & s & 0 & 0 \end{pmatrix} \right\}$$

That makes only three.....

Notice: $\varepsilon_1 - \varepsilon_2$ and $\varepsilon_1 + \varepsilon_2$ are conjugate by $\sigma = diag(1,1,1,-1) \in O(2,2)$!

$$H_3 = \sigma H_1 \sigma^{-1} \simeq \mathbb{C}^\times,\ \mathfrak{h}_3 = \left\{ X(x,r) = \begin{pmatrix} 0 & x & r & 0 \\ -x & 0 & 0 & -r \\ r & 0 & 0 & -x \\ 0 & -r & x & 0 \end{pmatrix} \right\}$$

Weyl groups for SO(2,2)

Fundamental CSG: $W(G, H_0)$ is an abelian 2-group of rank 1;

$\widehat{H_0}/W(G, H_0) : (k_1, k_2) \in \mathbb{Z}^2$ with $(k_1, k_2) \sim (-k_1, -k_2)$

For H_1 (or H_3), if $\varepsilon(X) = ix$, and $f(X) = r$, then $\Delta_i = \{\pm 2\varepsilon\}$ (noncompact), $\Delta_r = \{\pm 2f\}$, and there are no complex roots.

W_i acts by inversion on $SO(2)$, and the element $diag(1, -1, 1, -1) \in K$ has the same effect; i. e., $\mathcal{A}(H_1)$ has rank 1. W_r acts by inversion on $\mathbb{R}_+^\times$.

$\widehat{H_1}/W(G, H_1) : (k, \nu) \in \mathbb{Z} \times \mathbb{C}$ with $(k, \nu) \sim (\pm k, \pm \nu)$.

For the split Cartan, $W(G, H_2) = W_r$ the Weyl group of type $A_1 \times A_1$.

$\widehat{H_2}/W(G, H_2) : (\epsilon_1, \epsilon_2, \nu_1, \nu_2) \in (\mathbb{Z}/2\mathbb{Z})^2 \times \mathbb{C}^2$,
$(\epsilon_1, \epsilon_2, \nu_1, \nu_2) \sim (\epsilon_1, \epsilon_2, -\nu_1, -\nu_2) \sim (\epsilon_2, \epsilon_1, \nu_2, \nu_1) \sim (\epsilon_2, \epsilon_1, \nu_2, \nu_1)$

A Word About O(2,2)

In $O\,(2,2)$, we have only three CSGs, because $\varepsilon_1 - \varepsilon_2$ and $\varepsilon_1 + \varepsilon_2$ are conjugate by $\sigma = diag(1,1,1,-1)$.

Notice that this reflection is NOT in W_i! So $O(2,2)$ is not "nice" as in the Theorem.

Fundamental CSG: $W(G,H_0)$ is an abelian 2-group of rank 2;

$\widehat{H_0}/W(G,H_0) : (k_1,k_2) \in \mathbb{Z}^2$ with $(k_1,k_2) \sim (\pm k_1, \pm k_2)$

For H_1, $W(G,H_1)$ and the characters are as for $SO(2,2)$.

For the split Cartan, $W(G,H_2) \simeq W_r \rtimes \langle\sigma\rangle$, and
$(\epsilon_1,\epsilon_2,\nu_1,\nu_2) \sim (\epsilon_1,\epsilon_2,\pm\nu_1,\pm\nu_2) \sim (\epsilon_2,\epsilon_1,\pm\nu_2,\pm\nu_1)$

O(p,q) versus SO(p,q)

The group $O(p,q)$ is not "nice" (not in Harish-Chandra's class). However, $SO(p,q)$ is.

Some theorems only apply directly to "nice" groups.
`atlas` only deals with nice groups.

For $G = O(p,q)$, do the analysis for $SO(p,q)$ first.

If $p+q$ is odd, then $O(p,q) \simeq SO(p,q) \times \{\pm Id\}$.

For $O(p,q)$ with $p+q$ even, check what conjugation by diagonal elements does, as we did for $O(2,2)$.

CSGs in G=SO(p,q)

Recall that if $m \leq p, q$ then $GL(m, \mathbb{R}) \hookrightarrow SO(m, m) \hookrightarrow SO(p, q)$.

The CSGs of $GL(m, \mathbb{R})$ then extend to CSGs of $SO(p, q)$, by adding $SO(2)$ factors (and all of them may be obtained that way).

We get CSGs of the form

$$H_{r,t} = \left(\mathbb{C}^\times\right)^r \times \left(\mathbb{R}^\times\right)^t \times SO(2)^{n-2r-t}$$

Here $n = \left\lfloor \frac{p+q}{2} \right\rfloor$, the rank of G, and r and t integers such that:

- $2r + t \leq \min\{p, q\}$
- t is even if both p and q are, and t is odd of both p and q are
- If $2r = p = q$ then there are two nonconjugate CSGs that are isomorphic to $\left(\mathbb{C}^\times\right)^r$ (they become conjugate in $O(2r, 2r)$).

G=SO(7,3)

For example:

$G = SO(7,3)$ has rank 5.

There are three CSGs, corresponding to $(r,t) = (0,1)$, $(0,3)$, and $(1,1)$.

$H_0 \cong \mathbb{R}^\times \times SO(2)^4$

$H_1 \cong (\mathbb{R}^\times)^3 \times SO(2)^2$

$H_2 \cong \mathbb{C}^\times \times \mathbb{R}^\times \times SO(2)^2$

Real Weyl Groups: Exercise!

The End: Thank You!